**Curves & Curvature**

An embedded closed plane curve is a smooth map $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ where $\mathbb{R}^2$ is the plane, and

- $\gamma(0) = \gamma(1)$
- $\gamma'$ never vanishes
- $\gamma$ never crosses itself

The length of $\gamma$ is understood via its arclength parameter $s$, defined by

$$s(u) = \int_0^u \sqrt{|\gamma'|} \, du.$$  

The arclength induces a natural derivative

$$\partial_s = \frac{1}{\sqrt{|\gamma'|}} \partial_u.$$  

The curvature vector of $\gamma$ is the second derivative of $\gamma$ with respect to arclength:

$$\vec{k} = \partial_s^2 \gamma.$$  

For any $s$, $\vec{k}(s)$ is a vector pointing toward the centre of the osculating circle to $\gamma(s)$. The length of the curvature vector is the reciprocal of the radius of the osculating circle.

**Flows of Curves**

A curve flow is a smoothly-varying family of plane curves $\eta : [0, 1] \times [0, T) \rightarrow \mathbb{R}^2$, where for each $t$ the curve $\gamma_t$ defined by

$$\gamma_t(u) = \eta(u, t)$$

is smooth and embedded. The flow $\eta$ evolves according to

$$\partial_t \eta = F(\vec{k})$$

where $F$ is a transformation of the curvature vector $\vec{k}$. If $F$ is the identity, then the family $\eta$ is called a curve shortening flow.

**The Curve Shortening Flow**

Matt Grayson [1] proved in 1987 that any embedded, closed plane curve will become strictly convex under curve shortening flow. Michael Gage and Richard Hamilton [2] proved that after it becomes convex it shrinks to a point, becoming more and more round as it does so.

The example curve above (simulation courtesy of Sigurd Angenent) is not initially convex. The flow first pulls in tendrils with high curvature, and then works on removing all concave regions. We find that the curve is convex after approximately 0.225 seconds. The area enclosed by the curve decreases at the rate of $2\pi$ per second, and since the example curve above has area $\pi$, it has completely vanished after 0.5 seconds.

In this course students will learn fundamental techniques applicable to large families of curve flows through detailed presentation of the original arguments of [1, 2].

**Can you imagine...**

...how this curve contracts smoothly to a round point? (Art courtesy of Robert Bosch.)

**References**
